Fixed Point Theorem of Self mapping in a Complete 2-Banach Space

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Abstract: **This paper deals with few common fixed point theorems of self mapping in complete 2-Banach space which generalize the results of Som (2005), Amalendu Choudhury and T. Som (2011) and Mukti Gangopadhyay, Mantu Saha & A.P. Baisnab (2009).**

Keywords: **2-Banach Space, Convergent sequence, Cauchy sequence, fixed point.**

1. INTRODUCTION

The concept of 2-Banach space and some basic fixed point results are given by Gahler in 1960. In this paper I find some results in fixed point theory in a 2-Banach space by working with self mapping in a complete 2-Banach space.

2. PRELIMINARIES

Definition 2.1 :

Let X be a real linear space and $\|.,.\|$ be a non-negative real valued function defined on X satisfying the following conditions.

(i) $\|x, y.\| = 0$ if and only if x and y are linearly independent.

(ii) $\|x, y\| = \|y, x\|$ for all x,y∈X

(iii) $\|x, ay\| = |a|\|x, y\|$, a being real, for all x,y∈X

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all x,y,z∈X

Then $\|.,.\|$ is called a 2-norm and the pair (X, $\|.,.\|$) is called a linear 2-normed space. So, a 2-norm $\|x, y\|$ always satisfies $\|x, y + ax\| = \|x, y\|$ for all x,y∈X and all scalars a.

Definition 2.2:

A sequence {xn} in a 2-normed linear space (X, $\|.,.\|$) is said to be convergent to an element xεX if $\|x_n - x, a\| = 0$ *as* $n \to \propto$ *and for all* $a \in X$.

Definition 2.3:

If the sequence $\{x_n\}$ converges to x then $\|x_n - a, b\| = \|x - a, b\|$ as n→ ∞ & *for all* $a, b \in X$.

Definition 2.4:

A sequence $\{x_n\}$ in a 2-normed linear space (X, $\|.,.\|$) is a Cauchy sequence if $\|x_m - x_n, a\| = 0$ as m.n→ ∞ and for all $a \in X$.

Definition 2.5:

If a sequence is convergent in a 2-normed linear space, then it is a Cauchy sequence.

Definition 2.6:

A 2-normed linear space (X, $\|.,.\|$) is said to be complete if every Cauchy sequence in X is convergent.

3. PROPOSITIONS

3.1 : If a sequence $\{x_n\}$ in a 2-normed linear space converges to x then every subsequence of $\{x_n\}$ also converges to the same limit x.

3.2 : Limit of a sequence in a 2-normed linear space if it exists is unique.

4. MAIN RESULTS

Theorem 4.1 : Let (X, $\|.,.\|$) be a complete 2-normed linear space. Let T be a self map on X satisfying conditions:

$$\|T^i(x) - T^i(y), a\| \le \beta_i[\|y - Ty, a\| + \|y - Tx, a\|] + \gamma_i[\|x - y, a\|] \text{ for all x,y,a} \in X. \ldots(1)$$

Let $\beta_i \ge 0, 0 \le \gamma_i \le 1$ $(i = 1,2,3, \ldots)$ with $\sum_{i=1}^{\infty}(\beta_i + \gamma_i) < \infty$.

Then T has a unique fixed point in X.

Proof : Let any x$\in X$.

Let $x_n = T^n x_0$

Then $\|Tx_0 - T^2x_0, a\| = \|Tx_0 - TTx_0, a\|$

$$\le \beta_1[\|Tx_0 - TTx_0, a\| + \|Tx_0 - Tx_0, a\|] + \gamma_1[\|x_0 - Tx_0, a\|] \quad \text{(From (1))}$$

$$= \beta_1[\|Tx_0 - T^2x_0, a\| + \|x_1 - x_1, a\|] + \gamma_1[\|x_0 - Tx_0, a\|]$$

$$= \beta_1[\|Tx_0 - T^2x_0, a\|] + \gamma_1[\|x_0 - Tx_0, a\|]$$

$$= \beta_1[\|Tx_0 - T^2x_0, a\|] + \gamma_1[\|x_0 - Tx_0, a\|]$$

$$\|Tx_0 - T^2x_0, a\| \leq \beta_1[\|Tx_0 - T^2x_0, a\|] + \gamma_1[\|x_0 - Tx_0, a\|]$$

$$\|Tx_0 - T^2x_0, a\| - \beta_1[\|Tx_0 - T^2x_0, a\|] \leq \gamma_1[\|x_0 - Tx_0, a\|]$$

$$\|Tx_0 - T^2x_0, a\|[1 - \beta_1] \leq \gamma_1[\|x_0 - Tx_0, a\|]$$

$$\|Tx_0 - T^2x_0, a\| \leq \frac{\gamma_1}{[1-\beta_1]}[\|x_0 - Tx_0, a\|] \quad \ldots.(2)$$

Now, $\|x_n - x_{n+1}, a\| = \|T^n x_0 - T^{n+1}x_0, a\|$

$$=\|T^n x_0 - T^n T x_0, a\|$$

$$\leq \beta_n[\|Tx_0 - TTx_0, a\| + \|Tx_0 - Tx_0, a\|] + \gamma_n\|x_0 - Tx_0, a\|$$

$$= \beta_n\|Tx_0 - T^2x_0, a\| + \gamma_n\|x_0, Tx_0, a\|$$

$$\|x_n - x_{n+1}, a\| \leq \beta_n\|Tx_0 - T^2x_0, a\| + \gamma_n\|x_0 - Tx_0, a\|$$

$$\leq \beta_n \frac{\gamma_1}{1-\beta_1}\|x_0 - Tx_0, a\| + \gamma_n\|x_0 - Tx_0, a\| \quad [From\ (2)]$$

$$= \left(\beta_n \frac{\gamma_1}{1-\beta_1} + \gamma_n\right)\|x_0 - Tx_0, a\|$$

$$\therefore \|x_n - x_{n+1}, a\| \leq \left(\beta_n \frac{\gamma_1}{1-\beta_1} + \gamma_n\right)\|x_0 - Tx_0, a\| \quad \ldots.(3)$$

$$\|x_n - x_{n+2}, a\| = \|x_n - x_{n+1} + x_{n+1} - x_{n+2}, a\|$$

$$\leq \|x_n - x_{n+1}, a\| + \|x_{n+1} - x_{n+2}, a\|$$

$$= \sum_{k=0}^{1}\|x_{n+k} - x_{n+k+1}, a\|$$

$$\|x_n - x_{n+3}, a\| = \|x_n - x_{n+1} + x_{n+1} - x_{n+3}, a\|$$

$$\leq \|x_n - x_{n+1}, a\| + \|x_{n+1} - x_{n+3}, a\|$$

$$= \|x_n - x_{n+1}, a\| + \|x_{n+1} - x_{n+2} + x_{n+2} - x_{n+3}, a\|$$

$$\leq \|x_n - x_{n+1}, a\| + \|x_{n+1} - x_{n+2}, a\| + \|x_{n+2} - x_{n+3}, a\|$$

$$= \sum_{k=0}^{2}\|x_{n+k} - x_{n+k+1}, a\|$$

So for any positive integer p,

$$\|x_n - x_{n+p}, a\| \leq \sum_{k=0}^{p-1} \|x_{n+k} - x_{n+k+1}, a\|$$

$$\|x_n - x_{n+p}, a\| = \|x_{n+p} - x_n, a\|$$

$$\leq \sum_{k=0}^{p-1} \|x_{n+k} - x_{n+k+1}, a\|$$

$$\leq \left\{\sum_{k=0}^{p-1} \left(\frac{\gamma_1}{1-\beta_1}\right) \beta_{n+k} + \gamma_{n+k}\right\} \|x_0 - Tx_0, a\| \quad From\ (3)$$

$$= \left(\frac{\gamma_1}{1-\beta_1}\right) \left\{\sum_{k=0}^{p-1} \beta_{n+k} + \sum_{k=0}^{p-1} \gamma_{n+k}\right\} \|x_0 - Tx_0, a\|$$

Since $\sum_n(\beta_n + \gamma_n) < \infty$, $\|x_{n+p} - x_n, a\| \to 0\ as\ n \to \infty$

Since, $\{x_n\}$ is a Cauchy sequence in X and X is complete, $\|x_n - u, a\| = 0\ as\ n \to \infty$

Again $\|x_{n+1} - Tu, a\| = \|T^{n+1}x_0 - Tu, a\|$

$$= \|TT^n x_0 - Tu, a\|$$

$$\leq \beta_1[\|u - Tu, a\| + \|u - TT^n x_0, a\|] + \gamma_1[\|T^n x_0 - u, a\|] \quad [From\ (1)]$$

$$= \beta_1[\|u - Tu, a\| + \|u - T^{n+1}x_0, a\|] + \gamma_1[\|T^n x_0 - u, a\|]$$

$$= \beta_1[\|u - Tu, a\| + \|u - x_{n+1}, a\|] + \gamma_1[\|x_n - u, a\|]$$

$$\|x_{n+1} - Tu, a\| \leq \beta_1[\|u - Tu, a\| + \|u - x_{n+1}, a\|] + \gamma_1[\|x_n - u, a\|]$$

Taking the limit as $n \to \infty$

$$\|x_n - u. a\| \to 0\ \&\ \|u - x_{n+1}, a\| \to 0$$

$$\therefore \lim_{n\to\infty} \|x_{n+1} - Tu, a\| \leq \lim_{n\to\infty} \beta_1 \|u - Tu, a\|$$

i.e., $\|u - Tu, a\| \leq \beta_1 \|u - Tu, a\|$

$$0 \leq \beta_1 \|u - Tu, a\| - \|u - Tu, a\|$$

$$= (\beta_1 - 1) \|u - Tu, a\|$$

$$\therefore (\beta_1 - 1) \|u - Tu, a\| \geq 0$$

Since, $0 \leq \beta_i \leq 1, \|u - Tu, a\| = 0$

$$\therefore Tu = u$$

Thus u is a fixed point of T.

To prove the uniqueness.

Let u and v be two fixed points of T.

$$\therefore \|u - v, a\| = \|Tu - Tv, a\|$$

$$\leq \beta_1[\|v - Tv, a\| + \|v - Tu, a\|] + \gamma_1\|u - v, a\|$$

$$= \beta_1\|v - Tv, a\| + \beta_1\|v - Tu, a\| + \gamma_1\|u - v, a\|$$

$$\leq \gamma_1\|u - v, a\|$$

$$\therefore \|u - v, a\| \leq \gamma_1\|u - v, a\|$$

Since, $0 \leq \gamma_i \leq 1, u = v.$

∴ The fixed point of T is unique.

Theorem 4.2:

Let (X, ||. , . ||) be a complete 2-normed linear space. Let T be a self-mapping on X satisfying the conditions:

$$\|T^i x - T^i y, a\| \leq \beta_i[\|y - Ty, a\| + \|y - Tx, a\|] + \gamma_i\|x - y, a\| \; for\; all\; x, y, a \in X \; and$$

$$0 \leq \beta_i, \gamma_i < 1 \; (i = 1,2,3, \ldots with \sum_n (\beta_n + \gamma_n) < \infty.$$

If for some $x \in X, \{T^n(x_n)\}$ has a subsequence $\{T^{n_k}(x)\}$ with $\lim_{k\to\infty} \{T^{n_k}(x)\} = u \in X$. Then u is a unique fixed point of T.

Proof : We have for $u, x, a \in X$

$$\|u - Tu, a\| = \|u - T^{n_k+1}x + T^{n_k+1}x - Tu, a\|$$

$$\leq \|u - T^{n_k+1}x, a\| + \|T^{n_k+1} - Tu, a\|$$

$$\|T^{n_k+1}x - Tu, a\| = \|TT^{n_k}x - Tu, a\|$$

$$\leq \beta_1[\|u - Tu, a\| + \|u - T^{n_k+1}x, a\|] + \gamma_1\|T^{n_k}x - u, a\|$$

$$= \beta_1\|u - Tu, a\| + \beta_1\|u - T^{n_k+1}x, a\| + \gamma_1\|T^{n_k}x - u, a\|$$

Taking the limit as $k \to \infty$

$$\|u - Tu, a\| \leq \beta_1\|u - Tu, a\|$$

$$0 \leq \beta_1\|u - Tu, a\| - \|u - Tu, a\|$$

$$0 \leq (\beta_1 - 1)\|u - Tu, a\|$$

$$(\beta_1 - 1)\|u - Tu, a\| \geq 0$$

Since, $0 \leq \beta_i \leq 1, \|u - Tu, a\| = 0$

$$\therefore u = Tu$$

Hence u is a fixed point of T.

Clearly, the fixed point of T is unique.

REFERENCES

[1] Iseki K (1976): Fixed Point theorems in Banach Spaces, Math. Sem. Notes, Kobe Univ. pp. 211-213.

[2] Khan M S., Khan M.D., (1993) : Involutions with fixed points in 2-Banach Spaces, Int. J. Math & Math. Sci., 16, pp. 429-434.

[3] Som. T., (2005) : Some fixed point results in 2-Banach Space, International Jour. Math. Sci. 4(2), pp. 323-328.

[4] V.H. Badshah, Rekha Jain and Saurabh Jain : Common fixed point theorem for four mappings in Complete Spaces, Journal of Indian Acad. Math. Vol.33, No.1(2011) pp.97-103.

[5] Amalendu Choudhury and T. Som : 2-Banach space and some fixed point results, Journal of Indian Acad. Math. Vol.33, No.2(2011) pp 411-418.

[6] Neeta Tiwari, Tamakant Bhardway and R.P. Dubey : Common Fixed Point Theorem for Banach Space for Four Mapping, International Journal of Theoretical and Applied Sciences 5(1): 58-66 (2013).